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The Approximation of Fixed Points of Compositions of Nonexpansive Mappings in Hilbert Space

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Determining fixed points of nonexpansive mappings is a frequent problem in mathematics and physical sciences. An algorithm for finding common fixed points of nonexpansive mappings in Hilbert space, essentially due to Halpern, is analyzed. The main theorem extends Wittmann's recent work and partially generalizes a result by Lions. Algorithms of this kind have been applied to the convex feasibility problem.

1. INTRODUCTION AND NOTATION

Numerous problems in mathematics and physical sciences can be recast in terms of a *fixed point problem for nonexpansive mappings*. For instance, if the nonexpansive mappings are projections onto some closed convex sets, then the fixed point problem becomes the famous *convex feasibility problem*. Due to the practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be a flourishing topic of interest in fixed point theory.

Throughout the paper, we assume that

$$H \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle \text{ and induced norm } \|\cdot\|.$$

Suppose $T_1, \dots, T_N$ are nonexpansive self-mappings of some closed convex subset C of H (recall that a self-mapping T of C is *nonexpansive*, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$). We aim to solve the *fixed point problem for nonexpansive mappings*: find a common fixed point, i.e., find a

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point in $\bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$, where $\operatorname{Fix}(T_i) := \{x \in C: x = T_i x\}$ denotes the set of *fixed points of* T_i. If each T_i is the *projection* P_{C_i} *onto some closed convex nonempty set* C_i (see Fact 2.4), then $\operatorname{Fix}(T_i) = C_i$ and we thus obtain the *convex feasibility problem*: find a point in the intersection $\bigcap_{i=1}^{N} C_i$. (See [1, 6] and the references therein for more on convex feasibility problems.)

The most straightforward attempt to solve the fixed point problem for nonexpansive mappings is to iterate the mappings cyclically:

$$\begin{aligned} C \ni x_0 \mapsto x_1 := T_1 x_0 \mapsto \cdots \mapsto x_N := T_N x_{N-1} \\ \mapsto x_{N+1} := T_1 x_N \mapsto \cdots . \end{aligned} \tag{1}$$

For convenience, we set $T_n := T_{n \bmod N}$, where we let the mod N function take values in $\{1, \ldots, N\}$. Then we can rewrite (1) more compactly:

$$x_{n+1} := T_{n+1} x_n, \qquad \text{for all } n \geq 0; \quad x_0 \in C. \tag{1}$$

Unfortunately, algorithm (1) can fail to produce a norm convergent sequence (x_n) even if $N = 1$ and T_1 is *firmly nonexpansive* (a self-mapping T of C is called *firmly nonexpansive*, if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$, for all $x, y \in C$; see [9, 14, 16]), since Genel and Lindenstrauss [8] supplied an example where (x_n) converges only weakly. (Iteration (1) is then of type *Krasnoselski-Mann*; see Borwein *et al*. [4] for more.) If each nonexpansive mapping is a projection onto a closed convex nonempty set, then algorithm (1) becomes the well-known *method of cyclic projections*. However, even for $N = 2$ it is still not known whether or not convergence of the sequence (x_n) produced by the method of cyclic projections can actually be only weak! (Some positive results and more on the method of cyclic projections can be found in [2, 3] and the references therein.)

In view of the immensely successful Banach's contraction mapping principle, the attempt of approximating each nonexpansive self-mapping by Banach contractions seems very promising: indeed, for a sequence (λ_n) in $]0, 1[$ converging to 1, one obtains the following modified version of (1):

$$\begin{aligned} C \ni x_0 \mapsto x_1 := \lambda_1 a + (1 - \lambda_1) T_1 x_0 \mapsto \cdots \\ \mapsto x_N := \lambda_N a + (1 - \lambda_N) T_N x_{N-1} \\ \mapsto x_{N+1} := \lambda_{N+1} a + (1 - \lambda_{N+1}) T_1 x_N \mapsto \cdots ; \end{aligned} \tag{2}$$

or, more compactly,

$$x_{n+1} := \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad \text{for all } n \geq 0; \quad a, x_0 \in C. \tag{2}$$

In 1967, Halpern [11] suggested algorithm (2) for $N = 1$. Ten years later, Lions [12] investigated the general case. The restrictions they imposed on the sequence (λ_n) are, however, cumbersome to verify and exclude the "obvious" candidate $(1/(n+1))$. Recently Wittmann [15] extended the class of admissible sequences considerably for the original Halpern set up, i.e., when $N = 1$.

The objective of this paper is to improve and unify the results by Wittmann and Lions.

It turns out that algorithm (2) yields—under assumptions easier to verify than the assumptions suggested by Lions—a sequence (x_n) converging *in norm* to the common fixed point of $T_1, \ldots, T_N$ that is *nearest* to a. Our main result extends Wittmann's analysis of (2) for $N = 1$ and partially generalizes a result by Lions. In view of its attractive convergence property, algorithm (2) is well-suited for best approximation and convex feasibility problems.

The paper is organized as follows. Useful facts on projections and relevant classes of nonexpansive mappings are collected in Section 2. The third section contains our main theorem and a comparison to Lions' result.

Finally, we write B_H for the *unit ball* $\{x \in H: \|x\| \leq 1\}$ and I for the *identity mapping*. For sequences, the symbol $\to$ (resp. $\rightharpoonup$) indicates *norm* (resp. *weak*) convergence.

2. TOOLS

PROPOSITION 2.1. *Suppose* $(\lambda_k)_{k \geq 1}$ *is a sequence in* $[0, 1[$ *converging to* 0. *Then*

$$\sum_{k=1}^{\infty} \lambda_k = +\infty \Leftrightarrow \prod_{k=1}^{\infty} (1 - \lambda_k) := \lim_{n \to \infty} \prod_{k=1}^{n} (1 - \lambda_k) = 0.$$

Proof. We assume without loss of generality that $\lambda_k \leq 1/2$, for all k. Now $-x \geq \ln(1-x) \geq -2x$, for all $x \in [0, 1/2]$; thus

$$-\sum_{k=1}^{n} \lambda_k \geq \sum_{k=1}^{n} \ln(1 - \lambda_k) = \ln\left(\prod_{k=1}^{n} (1 - \lambda_k) \right) \geq -2 \sum_{k=1}^{n} \lambda_k,$$

and the result follows. ∎

Fact 2.2 (Goebel and Reich [10, Proposition 5.3]). Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset of H. Then $\operatorname{Fix}(T)$ is closed and convex.

Fact 2.3 (Opial's Demiclosedness Principle; see [13, Lemma 2]). Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset C of H. If (x_n) is a sequence in C converging weakly to x with $x_n - Tx_n \to 0$, then x is a fixed point of T.

Fact 2.4 (Projection; see [16, Lemma 1.1]). Suppose C is a closed convex nonempty subset of H with projection P_C. Then for every point $x \in H$, there exists a unique nearest point, denoted $P_C x$ and called *the projection of* x *onto* C: $\|x - P_C x\| \le \|x - c\|$, for all $c \in C$. Moreover, $P_C x$ is characterized by

$$P_C x \in C \qquad \text{and} \qquad \langle C - P_C x, x - P_C x\rangle \le 0.$$

DEFINITION 2.5 (Attracting Nonexpansive Mapping). We say that a nonexpansive self-mapping T of some closed convex subset C of H is *attracting*, if

$$\|Tx - f\| < \|x - f\|, \qquad \text{for all } f \in \operatorname{Fix}(T), x \in C \setminus \operatorname{Fix}(T).$$

The class of attracting nonexpansive mappings is relatively large and contains mappings which do not belong to any of the standard classes; see [1, Sec. 2]. Sufficient for our purpose are the following facts:

Fact 2.6 [1, Lemma 2.4, Corollary 2.5]. Suppose T is a firmly nonexpansive self-mapping of some closed convex nonempty subset of H. Then

$$(1 - \alpha)I + \alpha T \text{ is attracting}, \qquad \text{for all } \alpha \in]0, 2[.$$

In particular, if C is a closed convex nonempty subset of H, then the *relaxed projection* $(1 - \alpha)I + \alpha P_C$ is attracting, for all $\alpha \in]0, 2[$.

Fact 2.7 [1, Proposition 2.10]. Suppose $T_1, \dots, T_N$ are nonexpansive attracting self-mappings of some closed convex nonempty subset of H. Then $\operatorname{Fix}(T_N \cdots T_1) = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$, provided the latter set is nonempty. In particular, if $C_1, \dots, C_N$ are closed convex subsets of H with $\bigcap_{i=1}^N C_i \neq \varnothing$, and if $\alpha_1, \dots, \alpha_N$ are in $]0, 2[$, then

$$\operatorname{Fix}\Big(\big((1 - \alpha_N)I + \alpha_N P_{C_N}\big) \cdots \big((1 - \alpha_1)I + \alpha_1 P_{C_1}\big)\Big) = \bigcap_{i=1}^N C_i.$$

3. MAIN RESULT

Assumption on the Mappings. $T_1, \dots, T_N$ are nonexpansive self-mappings of some closed convex subset C of H with $F := \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ nonempty and

$$\begin{aligned} F = \operatorname{Fix}(T_N \cdots T_1) &= \operatorname{Fix}(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Assumption on the Parameters. $(\lambda_n)_{n \geq 1}$ is a *sequence of parameters* in $[0, 1[$ which satisfies the following:

(A1) $\lambda_n \to 0$.

(A2) $\prod_n (1 - \lambda_n) = 0$; equivalently (Proposition 2.1), $\sum_n \lambda_n = +\infty$.

(A3) $\sum_n |\lambda_n - \lambda_{n+N}| < +\infty$.

As in the Introduction, we set $T_n := T_{n \bmod N}$, where we let the mod N function take values in $\{1, \ldots, N\}$.

Given points a, x_0 in C, a sequence (x_n) is generated by

$$x_{n+1} := \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad \text{for all } n \geq 0; \tag{2}$$

we say (x_n) *has anchor a and starting point x_0*.

THEOREM 3.1. *If the above assumptions on the mappings and on the parameters hold, then the sequence generated by* (2) *converges in norm to* $P_F a$.

Proof. Fact 2.2 and Fact 2.4 ensure that the point $P_F a$ is well-defined. We now prove a particular case of the result.

Special Case. $x_0 = a$.

Claim. (1) $\|x_n - f\| \leq \|a - f\|$, for all $n \geq 0$ and every $f \in F$.

We proceed by induction on n. Fix $f \in F$. Clearly, (1) holds for $n = 0$. If $\|x_n - f\| \leq \|a - f\|$, then

$$\begin{aligned} \|x_{n+1} - f\| &\leq \lambda_{n+1}\|a - f\| + (1 - \lambda_{n+1})\|T_{n+1}x_n - f\| \\ &\leq \lambda_{n+1}\|a - f\| + (1 - \lambda_{n+1})\|x_n - f\| \\ &\leq \|a - f\|, \end{aligned}$$

as desired. It follows that:

(2) (x_n) is bounded (by (1)).

(3) $(T_{n+1}x_n)$ is bounded (by (2)).

(4) $x_{n+1} - T_{n+1}x_n \to 0$ (by (3)).

Claim. (5) $x_{n+N} - x_n \to 0$.

By (2) and (3), there is some constant $L \geq 0$ such that $\|x_{n+N} - x_n\|$, $\|a - T_{n+1}x_n\| \leq L$, for all $n \geq 0$. Hence, for all $n \geq 1$,

$$\begin{aligned} \|x_{n+N} - x_n\| &= \|(\lambda_{n+N} - \lambda_n)(a - T_n x_{n-1}) \\ &\qquad + (1 - \lambda_{n+N})(T_n x_{n+N-1} - T_n x_{n-1})\| \\ &\leq L|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\|. \end{aligned}$$

Thus inductively

$$\|x_{n+N} - x_n\| \le L \sum_{k=m+1}^{n} |\lambda_{k+N} - \lambda_k| + \|x_{m+N} - x_m\| \prod_{k=m+1}^{n} (1 - \lambda_{k+N}),$$

for all $n \ge m \ge 0$. Hence

$$\overline{\lim}_n \|x_{n+N} - x_n\| \le L \sum_{k=m+1}^{\infty} |\lambda_{k+N} - \lambda_k| + L \prod_{k=m+1}^{\infty} (1 - \lambda_{k+N}).$$

On the other hand, assumptions (A3) and (A2) imply $\lim_m \sum_{k=m+1}^{\infty} |\lambda_{k+N} - \lambda_k| = 0$ and $\lim_m \prod_{k=m+1}^{\infty}(1 - \lambda_k) = 0$. Altogether, by letting m tend to $+\infty$, we conclude $x_{n+N} - x_n \to 0$, as promised.

Claim. (6) $x_n - T_{n+N} \cdots T_{n+1} x_n \to 0$.

In view of (5)), it suffices to show that $x_{n+N} - T_{n+N} \cdots T_{n+1} x_n \to 0$. Now

$$x_{n+N} - T_{n+N} x_{n+N-1} \to 0,$$

by (4). Again by (4), $x_{n+N-1} - T_{n+N-1} x_{n+N-2} \to 0$; thus the nonexpansiveness of T_{n+N} implies

$$T_{n+N} x_{n+N-1} - T_{n+N} T_{n+N-1} x_{n+N-2} \to 0.$$

Similarly,

$$\begin{gathered} T_{n+N} T_{n+N-1} x_{n+N-2} - T_{n+N} T_{n+N-1} T_{n+N-2} x_{n+N-3} \to 0, \\ \vdots \\ T_{n+N} \cdots T_{n+2} x_{n+1} - T_{n+N} \cdots T_{n+1} x_n \to 0. \end{gathered}$$

Adding these N sequences yields (6).

Claim. (7) $\overline{\lim}_n \langle T_{n+1} x_n - P_F a, a - P_F a \rangle \le 0$.

Pick a subsequence $(x_{n'})$ of (x_n) such that $\lim_{n'} \langle T_{n'+1} x_{n'} - P_F a, a - P_F a \rangle = \overline{\lim}_n \langle T_{n+1} x_n - P_F a, a - P_F a \rangle$. We assume (after passing to another subsequence if necessary) that $n' + 1 \bmod N = i$, for some $i \in \{1, \dots, N\}$, and that $x_{n'+1} \rightharpoonup x$. By (6), $x_{n'+1} - T_{i+N} \cdots T_{i+1} x_{n'+1} \to 0$; hence the Demiclosedness Principle (Fact 2.3) implies $x \in \operatorname{Fix}(T_{i+N} \cdots T_{i+1}) = F$. Therefore, by (4) and Fact 2.4,

$$\begin{aligned} \overline{\lim}_n \langle T_{n+1} x_n - P_F a, a - P_F a \rangle &= \lim_{n'} \langle T_{n'+1} x_{n'} - P_F a, a - P_F a \rangle \\ &= \lim_{n'} \langle T_{n'+1} x_{n'} - x_{n'+1}, a - P_F a \rangle \\ &\quad + \lim_{n'} \langle x_{n'+1} - P_F a, a - P_F a \rangle \\ &= \langle x - P_F a, a - P_F a \rangle \\ &\le 0, \end{aligned}$$

as required.

Now fix an arbitrary $\epsilon > 0$ and get (because of (7) and (A1)) an n_ϵ such that

$$\langle T_{n+1}x_n - P_F a, a - P_F a\rangle \leq \epsilon \text{ and } \lambda_n \|a - P_F a\|^2 \leq \epsilon, \text{ for all } n \geq n_\epsilon.$$

Then

$$\begin{aligned}\|x_{n+1} - P_F a\|^2 &= \|\lambda_{n+1}(a - P_F a) + (1-\lambda_{n+1})(T_{n+1}x_n - P_F a)\|^2 \\ &= \lambda_{n+1}^2\|a - P_F a\|^2 + (1-\lambda_{n+1})^2\|T_{n+1}x_n - P_F a\|^2 \\ &\quad + 2\lambda_{n+1}(1-\lambda_{n+1})\langle a - P_F a, T_{n+1}x_n - P_F a\rangle \\ &\leq \lambda_{n+1}\epsilon + (1-\lambda_{n+1})\|x_n - P_F a\|^2 + 2\lambda_{n+1}\epsilon,\end{aligned}$$

and hence inductively

$$\|x_{n+1} - P_F a\|^2 \leq 3\epsilon + \|x_{n_\epsilon} - P_F a\|^2 \prod_{k=n_\epsilon+1}^{n} (1-\lambda_k), \qquad \text{for all } n \geq n_\epsilon.$$

Letting n tend to $+\infty$ gives $\overline{\lim}_n \|x_n - P_F a\|^2 \leq 3\epsilon$. Since ϵ was arbitrary, we conclude $x_n \to P_F a$; that is, the special case is verified.

General Case. x_0 is arbitrary.

Now (x_n) has anchor a and starting point x_0 (possibly different from a). Let (y_n) be the sequence with anchor a and starting point $y_0 := a$. On the one hand, by the special case,

$$y_n \to P_F a.$$

On the other hand, it is easily checked that

$$\|x_n - y_n\| \leq \|x_0 - y_0\| \prod_{k=1}^{n} (1-\lambda_k), \qquad \text{for all } n \geq 0.$$

Thus $x_n - y_n \to 0$ and altogether $x_n \to P_F a$. ∎

COROLLARY 3.2 (Wittmann [15, Theorem 2]). *Suppose T is a nonexpansive self-mapping of some closed convex nonempty subset C of H. Suppose further* $\mathrm{Fix}(T) \neq \varnothing$ *and* (λ_n) *is a sequence satisfying the assumptions on the parameters* (*with* $N = 1$). *Then the sequence* (x_n) *generated by*

$$x_0 \in C \text{ arbitrary}, \qquad x_{n+1} := \lambda_{n+1}x_0 + (1-\lambda_{n+1})Tx_n \ (n \geq 0)$$

converges in norm to $P_{\mathrm{Fix}(T)}x_0$.

Remarks 3.3. (1) The predecessor of Wittmann's result is Halpern [11, Theorem 4]; see also Browder [5].

(2) As examples with $N = 2$ and $T_1 = T_2 = -I$ show, *some assumption on the mappings is necessary*.

(3) In view of Fact 2.6 and Fact 2.7, the assumption on the mappings holds for the relatively large class of attracting nonexpansive mappings with common fixed points, in particular, for relaxed projections. It is clear that Theorem 3.1 and related results have applications to best approximation and convex feasibility problems: see, for instance, Combettes [7].

(4) Halpern already pointed out that *assumptions* (A1) *and* (A2) *are necessary*; see [11, Theorem 2].

(5) Assumption (A3) is relatively easy to check and holds in particular when (i) $\sum_n |\lambda_n - \lambda_{n+1}| < +\infty$; (ii) $(\lambda_{kN+i})_{k\geq 0}$ is decreasing, for all $i = 1, \dots, N$; or (iii) (λ_n) is decreasing. Remark 3.6.1 below shows that (A3) *is not necessary*.

(6) It would be interesting to know how the algorithm acts when F is empty. We conjecture that then $\lim_n \|x_n\| = +\infty$.

Remark 3.4. Dr. Mark Limber has performed numerous experiments with the algorithms (1) and (2) in the context of image reconstruction in medical imaging. The nonexpansive mappings involved were projections onto hyperplanes and onto the positive orthant. His conclusions can be summarized as follows: although the numerical results produced by algorithm (2) were different from those obtained by algorithm (1) (*the method of cyclic projections*), the corresponding pictures were indistinguishable. Hence the algorithms are similar with respect to performance and computational cost. The method of cyclic projections is slightly cheaper (no parameters to compute), whereas the algorithm (2) has attractive convergence properties (Theorem 3.1). So it is up to the user to choose the algorithm better suited for his or her problem.

Let us now compare Theorem 3.1 to the following result of Lions:

Fact 3.5 (Lions [12, Théorème 4]). Suppose $T_1, \dots, T_N$ are firmly nonexpansive self-mappings of some closed convex nonempty subset C of H with $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \varnothing$. Suppose further $(\lambda_n)_{n\geq 1}$ is a sequence in $[0, 1[$ which satisfies:

(L1) $\lambda_n \to 0$.

(L2) $\sum_k \lambda_{kN+i} = +\infty$, for all $i = 0, \dots, N-1$.

(L3) $\lim_k (\sum_{i=1}^N |\lambda_{kN+i} - \lambda_{(k-1)N+i}|)/(\sum_{i=1}^N \lambda_{kN+i})^2 = 0$.

Then the sequence (x_n) generated by (2) converges in norm to $P_F a$.

Remarks 3.6. (1) Clearly, (L2) is more restrictive than (A2). In general, (A3) and (L3) are independent, even when $N = 1$: if $\lambda_n := 1/(n+1)$, then (λ_n) satisfies (A3) and fails (L3). In contrast, if (λ_n) is given by $\lambda_{2n} := (n+1)^{-1/4}$ and $\lambda_{2n+1} := (n+1)^{-1/4} + (n+1)^{-1}$, then (L3) holds but (A3) does not.

(2) For sequences of parameters satisfying the easy-to-verify conditions of Remarks 3.3.5, Theorem 3.1 is much more powerful than Fact 3.5, because the former applies to a genuinely bigger class of mappings and the latter requires in addition (L3).

(3) The "most natural" choice of parameters, $\lambda_n := 1/(n+1)$,

- is computationally attractive (inexpensive evaluations),
- satisfies (A3) but fails (L3),
- has an interesting by-product, namely Eberlein's linear mean ergodic theorem (see [15]).

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REFERENCES

1. H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, to appear.
2. H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Anal.* **1** (1993), 185–212.
3. H. H. Bauschke, J. M. Borwein, and A. S. Lewis, The method of cyclic projections for closed convex sets in Hilbert space, Proceedings of the Special Session on Optimization and Nonlinear Analysis, Jerusalem, May 1995 (Y. Censor and S. Reich, Eds.). *Contemp. Math*, to appear.
4. J. M. Borwein, S. Reich, and I. Shafrir, Krasnoselski-Mann iterations in normed spaces, *Canad. Math. Bull.* **35** (1992), 21–28.
5. F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* **24** (1967), 82–90.
6. P. L. Combettes, The foundations of set theoretic estimation, *Proc. IEEE* **81** (1993), 182–208.
7. P. L. Combettes, Inconsistent signal feasibility problems: Least-squares solutions in a product space, *IEEE Trans. Signal Process.* **42** (1994) 2955–2966.
8. A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel J. Math.* **22** (1975), 81–86.
9. K. Goebel and W. A. Kirk, Topics in metric fixed point theory, *in* "Cambridge Studies in Advanced Mathematics," Vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.
10. K. Goebel and S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, *in* "Monographs and Textbooks in Pure and Applied Mathematics," Vol. 83, Dekker, New York, 1984.
11. B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* **73** (1967), 957–961.
12. P.-L. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Sér. A-B Paris* **284** (1977), 1357–1359.
13. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

14. R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), 877–898.
15. R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **58** (1992), 486–491.
16. E. H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, *In* "Contributions to Nonlinear Functional Analysis," (E. H. Zarantonello, Ed.), pp. 237–424, Academic Press, New York, 1971.